Exercise 24

Given functions $q(x) = \frac{1}{\sqrt{x}}$ and $h(x) = x^2 - 9$, state the domain of each of the following functions using interval notation:

(a) $\frac{q(x)}{h(x)}$

(b) $q(h(x))$

(c) $h(q(x))$

Solution

Part (a)

Compute the function $q(x)/h(x)$.

$$\frac{q(x)}{h(x)} = \frac{\frac{1}{\sqrt{x}}}{x^2 - 9}$$

$$= \frac{1}{\sqrt{x}(x^2 - 9)}$$

It's impossible to divide by zero, and it's impossible to take the square root of a negative number.

$$x \geq 0 \quad \text{and} \quad \sqrt{x}(x^2 - 9) \neq 0$$

$$x \geq 0 \quad \text{and} \quad \left(\sqrt{x} \neq 0 \quad \text{or} \quad x^2 - 9 \neq 0\right)$$

$$x \geq 0 \quad \text{and} \quad \left[x \neq 0 \quad \text{or} \quad (x+3)(x-3) \neq 0\right]$$

$$x \geq 0 \quad \text{and} \quad \left(x \neq 0 \quad \text{or} \quad x + 3 \neq 0 \quad \text{or} \quad x - 3 \neq 0\right)$$

$$x \geq 0 \quad \text{and} \quad \left(x \neq 0 \quad \text{or} \quad x \neq -3 \quad \text{or} \quad x \neq 3\right).$$

Therefore, the domain of $q(x)/h(x)$ in interval notation is $(0, 3) \cup (3, \infty)$.

Part (b)

Compute $q(h(x))$ by plugging the formula for $h(x)$ where x is in the formula for $q(x)$.

$$q(h(x)) = \frac{1}{\sqrt{x^2 - 9}}$$

It's impossible to divide by zero, and it's impossible to take the square root of a negative number.

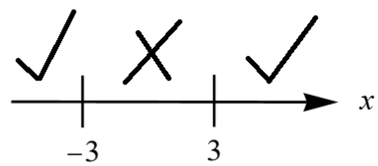
$$x^2 - 9 \geq 0 \quad \text{and} \quad \sqrt{x^2 - 9} \neq 0$$

$$(x+3)(x-3) \geq 0 \quad \text{and} \quad x^2 - 9 \neq 0$$

$$(x+3)(x-3) \geq 0 \quad \text{and} \quad x^2 \neq 9$$

$$(x+3)(x-3) \geq 0 \quad \text{and} \quad x \neq \pm 3$$

For the inequality on the left, the critical points are -3 and 3. Partition the number line at these numbers and test where the inequality is true.



Therefore, the domain of $q(h(x))$ in interval notation is $(-\infty, -3) \cup (3, \infty)$.

Part (c)

Compute $h(q(x))$ by plugging the formula for $q(x)$ where x is in the formula for $h(x)$.

$$m(q(x)) = \left(\frac{1}{\sqrt{x}}\right)^2 - 9 = \frac{1}{x} - 9$$

It's impossible to divide by zero, and it's impossible to take the square root of a negative number.

$$x \geq 0 \quad \text{and} \quad x \neq 0$$

Combine the two conditions.

$$x > 0$$

Therefore, the domain of $h(q(x))$ in interval notation is $(0, \infty)$.